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Physical Complexity of Classical and Quantum Objects and Their Dynamical Evolution From an Information-Theoretic Viewpoint

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Charles Bennett's measure of physical complexity for classical objects, namely logical-depth, is used to prove that a chaotic classical dynamical system is not physically complex. The natural measure of physical complexity for quantum objects, quantum logical-depth, is then introduced to prove that a chaotic quantum dynamical system too is not physically complex.

KEY WORDS: complexity; Information Theory; dynamical systems; Quantum Mechanics.

NOTATION

$\forall$	For every (universal quantificator)
$\forall - \mu - a.e.$	For $\mu - almost$ every
$\exists$	Exists (existential quantificator)
$\exists!$	Exists and is unique
$x = y$	x is equal to y
$x := y$	x is defined as y
$\neg p$	Negation of p
Σ	Binary alphabet $\{0, 1\}$
Σ_n	n-Letters' alphabet
$\Sigma_n^\star$	Strings on the alphabet Σ_n
Σ_n^∞	Sequences on the alphabet Σ_n
$\vec{x}$	String
$\|\vec{x}\|$	Length of the string $\vec{x}$
$<_l$	Lexicographical ordering on $\Sigma^\star$
string(n)	nth string in lexicographic order
$\bar{x}$	Sequence
$\cdot$	Concatenation operator

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x_n	nth digit of the string $\vec{x}$ or of the sequence $\bar{x}$
$\vec{x}(n)$	Prefix of length n of the string $\vec{x}$ or of the sequence $\bar{x}$
$\vec{x}^{\star}$	Canonical string of the string $\vec{x}$
$K(\vec{x})$	Plain Kolmogorov complexity of the string $\vec{x}$
$I(\vec{x})$	Algorithmic information of the string $\vec{x}$
$U(\vec{x})\downarrow$	U halts on input $\vec{x}$
$D_s(\vec{x})$	Logical-depth of $\vec{x}$ at significance level s
t-DEEP($\Sigma^{\star}$)	t-Deep strings of cbits
t-SHALLOW($\Sigma^{\star}$)	t-Shallow strings of cbits
REC-MAP($\mathbb{N}$, $\mathbb{N}$)	(Total) recursive functions over $\mathbb{N}$
$\leq_{\mathrm{T}}$	Turing reducibility
$\leq_{\mathrm{T}}^{\mathrm{P}}$	Polynomial time Turing reducibility
$\leq_{\mathrm{tt}}$	Truth-table reducibility
CHAITIN-RANDOM(Σ^{∞})	Martin–Löf–Solovay–Chaitin random sequences of cbits
HALTING(μ)	Halting set of the measure μ
μ_{Lebesgue}	Lebesgue measure
μ-RANDOM(Σ^{∞})	Martin Löf random sequences of cbits w.r.t. μ
$B(\bar{x})$	Brudno algorithmic entropy of the sequence $\bar{x}$
BRUDNO(Σ^{∞})	Brudno random sequences of cbits
STRONGLY-DEEP(Σ^{∞})	Strongly-deep sequences of cbits
STRONGLY-SHALLOW(Σ^{∞})	Strongly-shallow sequences of cbits
WEAKLY-DEEP(Σ^{∞})	Weakly-deep sequences of cbits
WEAKLY-SHALLOW(Σ^{∞})	Weakly-shallow sequences of cbits
$\mathcal{P}(X, \mu)$	(Finite, measurable) partitions of (X, μ)
$\preceq$	Coarse-graining relation on partitions
$A \bigvee B$	Coarsest refinement of A and B
h_{CDS}	Kolmogorov–Sinai entropy of CDS
ψ_A	Symbolic translator w.r.t. A
$\psi_A^{(k)}$	k-Point symbolic translator w.r.t. A
$\psi_A^{(\infty)}$	Orbit symbolic translator w.r.t. A
cb_{n_1,n_2}	Change of basis from n_1 to n_2
$B_{\mathrm{CDS}}(x)$	Brudno algorithmic entropy of x's orbit in CDS
$H(P)$	Shannon entropy of the distribution P
$L_{D,P}$	Average code-word length w.r.t. the code D and the distribution P
L_P	Minimal average code-word length w.r.t. the distribution P
$\mathcal{H}_2$	One-qubit's Hilbert space
$\mathcal{H}_2^{\otimes n}$	n-Qubits' Hilbert space
$\mathcal{E}_n$	Computational basis of $\mathcal{H}_2^{\otimes n}$

$\mathcal{H}_2^{\otimes\star}$	Hilbert space of qubits' strings
$\mathcal{E}_\star$	Computational basis of $\mathcal{H}_2^{\otimes\star}$
$\mathcal{H}_2^{\otimes\infty}$	Hilbert space of qubits' sequences
$\mathcal{E}_\infty$	Computational rigged-basis of $\mathcal{H}_2^{\otimes\infty}$
$\mathcal{B}(\mathcal{H})$	Bounded linear operators on $\mathcal{H}$
$\|\psi\rangle^\star$	Canonical program of $\|\psi\rangle$
$D_s(\|\psi\rangle)$	Logical depth of $\|\psi\rangle$ at significance level s
t-DEEP($\mathcal{H}_2^{\otimes\star}$)	t-Deep strings of qubits
t-SHALLOW($\mathcal{H}_2^{\otimes\star}$)	t-Shallow strings of qubits
$S(A)$	States over the noncommutative space A
card(A)	Cardinality of A
INN(A)	Inner automorphisms of A
$\text{card}_{\text{NC}}(A)$	Noncommutative cardinality of A
τ_{unbiased}	Unbiased noncommutative probability distribution
$\Sigma_{\text{NC}}^{\infty}$	Noncommutative space of qubits' sequences
R	Hyperfinite II_1 factor
RANDOM($\Sigma_{\text{NC}}^{\infty}$)	Random sequences of qubits
$\leq_{\text{tt}}^{\text{Q}}$	Quantum truth-table reducibility
WEAKLY-DEEPWEAKLY-DEEP($\Sigma_{\text{NC}}^{\infty}$)	Weakly-deep sequences of qubits
WEAKLY-SHALLOW($\Sigma_{\text{NC}}^{\infty}$)	Weakly-shallow sequences of qubits
$\mathcal{L}_{\text{RANDOMNESS}}^{\text{NC}}(A, \omega)$	Laws of randomness of (A, ω)
ω-RANDOM($\Sigma_{\text{NC}}^{\infty}$)	Random sequences of qubits w.r.t. ω
$h_{d.e.}$(QDS)	Dynamical entropy of the quantum dynamical system QDS
i.e.	id est
e.g.	Exempli gratia

1. INTRODUCTION: THE SHALLOWNESS OF RANDOM OBJECTS

Despite denoting it with the term *complexity*, Andreǐ Nikolaevich Kolmogorov (1992, 1993a,b,c; American Mathematical Society, 2000) introduced the notion denoted nowadays by the school of Paul Vitanyi (1997) as *plain-Kolmogorov-complexity* (that I will denote with the letter K from here and beyond) in order of obtaining an intrinsic measure of the *amount of information* of that object and not as a measure of the *amount of physical complexity* of that object.

That the *amount of information* and the *amount of physical complexity* of an object are two completely different concepts became further clear after the introduction by Gregory Joseph Chaitin of the notion denoted nowadays by the school of Paul Vitanyi (1997) as *prefix-Kolmogorov-complexity* and denoted by the school of Chaitin and Cristian S. Calude simply as *algorithmic information* (Calude, 2002) (and that I will denote with the letter I from here and beyond) and the induced notion of *algorithmic-randomness*:

An algorithmically random object has a very high algorithmic information but is certainly not physically complex.

Such a simple consideration is indeed sufficient to infer that *algorithmic information* can in no way be seen as a measure of physical complexity.

A natural measure of physical complexity within the framework of Algorithmic Information Theory, the *logical depth*, was later introduced by Gregory Chaitin and Charles Bennett (Bennett, 1988), constituting what is nowadays generally considered as the *algorithmic information theoretic viewpoint* as to physical complexity, although some author can still be found who not only ignores that, as it was clearly realized by Brudno himself (1978, 1983; Segre, 2002), the *chaoticity* of a dynamical system (defined as the strict positivity of its Kolmogorov–Sinai entropy) is equivalent to its *weak algorithmic chaoticity* (defined as the condition that almost all the trajectories, symbolically codified, are *Brudno-algorithmically random*) but is weaker than its *strong algorithmic chaoticity* (defined as the condition that almost all the trajectories, symbolically codified, are *Martin–Löf–Solovay–Chaitin-algorithmically random*), but uses the notions of *chaoticity* and *complexity* as if they were synonymous, a thing obviously false since, as we have seen, the (weak) algorithmic randomness of almost all the trajectories of a chaotic dynamical system implies exactly the opposite, namely that its trajectories are not *complex* at all.

Indeed it is natural to define *complex* a dynamical system such that almost all its trajectories, symbolically codified, are *logical deep*.

So, despite the still common fashion to adopt the terms *chaoticity* and *complexity* as synonymous, one has that that every chaotical dynamical system is shallow, as I will show in sections 2 and 3.

The key point of such an issue is so important to deserve a further repetition with the own words of Charles Bennett (1988) illustrating the physical meaning of the notion of *logical depth* and the reason why it is a good measure of physical complexity:

> The notion of logical-depth developed in the present paper was first described in Chaitin (1977), and at greater length in Bennett (1982) and Bennett (1985); similar notions have been independently introduced by Adleman (1979) ("potential"), (Levin and V'Jugin, 1977) ("incomplete sequence"), (Levin, 1984) ("hitting time") and Koppel, this volume ("sophistication"). See also Wolfram's work on "computational irreducibility" (Wolfram, 1985) and Hartmanis' work on time- and space-bounded algorithmic information (Hartmanis, 1983).
>
> We propose depth as a formal measure of value. From the earliest days of information theory it has been appreciated that information per se is not a good measure of message

> value. For example a typical sequence of coin tosses has high information content but little value; an ephemeresis, giving the positions of the moon and planets every day for a hundred years, has no more information than the equations of motions and initial conditions from which it was calculated, but saves its owner the effort of recalculating these positions. The value of a message thus appears to reside not in its information (its absolutely unpredictable parts), nor in its obvious redudancy (verbatim repetitions, unequal digit frequencies), but rather in what might be called its buried redudancy—parts predictable only with difficulty, things the receiver could in principle have figured out without being told, but only at considerable cost in money, time or computation. In other words the value of a message is the amount of mathematical or other work plausibly done by its originator, which its receiver is saved from having to repeat.

The quantum analogue of such a notion, i.e., *quantum logical depth*, is introduced in section 4.

The definition of the *physical complexity* of a quantum dynamical system is then introduced in section 5, where it is shown that in the quantum case, as in the classical case, a physically complex dynamical system is not chaotic.

2. DEFINITION OF THE PHYSICAL COMPLEXITY OF STRINGS AND SEQUENCES OF CBITS

I will follow from here on the notation of my PhD thesis (Segre, 2002); consequentially, given the binary alphabet $\Sigma := \{0, 1\}$, I will denote by $\Sigma^\star$ the set of all the strings on Σ (i.e., the set of all the strings of cbits), by Σ^∞ the set of all the sequences on Σ (i.e., the set of all the sequences of cbits) and by CHAITIN–RANDOM(Σ^∞) its subset consisting of all the Martin–Löf–Solovay–Chaitin random sequences of cbits.

I will furthermore denote strings by an upper arrow and sequences by an upper bar, so that I will talk about the string $\vec{x} \in \Sigma^\star$ and the sequence $\bar{x} \in \Sigma^\infty$; $|\vec{x}|$ will denote the length of the string $\vec{x}$, x_n will denote the nth digit of the string $\vec{x}$ or of the sequence $\bar{x}$ while $\vec{x}_n$ will denote its prefix of length n.

I will, finally, denote by $<_l$ the lexicographical-ordering relation over $\Sigma^\star$ and by string(n) the nth string in such an ordering.

Fixed once and for all a universal Chaitin computer U, let us recall the following basic notions.

Given a string $\vec{x} \in \Sigma^\star$ and a natural number $n \in \mathbb{N}$:

Definition 2.1. Canonical program of $\vec{x}$:

$$\vec{x}^\star := \min_{<_l}\{\vec{y} \in \Sigma^\star : U(\vec{y}) = \vec{x}\} \tag{2.1}$$

Definition 2.2. $\vec{x}$ is n-compressible:

$$|\vec{x}^\star| \leq |\vec{x}| - n \tag{2.2}$$

Definition 2.3. $\vec{x}$ is n-incompressible:

$$|\vec{x}^{\star}| > |\vec{x}| - n \tag{2.3}$$

Definition 2.4. Halting time of the computation with input $\vec{x}$:

$$T(\vec{x}) := \begin{cases} \text{number of computational steps after which } U \text{ halts on input } \vec{x} & \text{,if } U(\vec{x}) = \downarrow \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.4}$$

We have at last all the ingredients required to introduce the notion of *logical depth* as to strings.

Given a string $\vec{x} \in \Sigma^{\star}$ and two natural number $s, t \in \mathbb{N}$:

Definition 2.5. Logical depth of $\vec{x}$ at significance level s:

$$D_s(\vec{x}) := \min\{T(\vec{y}) : U(\vec{y}) = \vec{x}, \vec{y} \text{ } s\text{-incompressible}\} \tag{2.5}$$

Definition 2.6. $\vec{x}$ is t-deep at significance level s:

$$D_s(\vec{x}) > t \tag{2.6}$$

Definition 2.7. $\vec{x}$ is t-shallow at significance level s:

$$D_s(\vec{x}) \leq t \tag{2.7}$$

I will denote the set of all the t-deep strings as t-DEEP $(\Sigma^{\star})$ and the set of all the t-shallow strings as t-SHALLOW $(\Sigma^{\star})$.

Exactly as it is impossible to give a sharp distinction among *Chaitin-random* and *regular* strings while it is possible to give a sharp distinction among *Martin–Löf–Solovay–Chaitin-random* and *regular* sequences, it is impossible to give a sharp distinction among *deep* and *shallow* strings while it is possible to give a sharp distinction among *deep* and *shallow* sequences.

Given a sequence $\bar{x} \in \Sigma^{\infty}$:

Definition 2.8. $\bar{x}$ is strongly deep:

$$\text{card}\{n \in \mathbb{N} : D_s(\vec{x}(n)) > f(n)\} < \aleph_0 \, \forall s \in \mathbb{N}, \forall f \in REC-MAP(\mathbb{N}, \mathbb{N}) \tag{2.8}$$

where, following once more the notation adopted in Segre (2002), *REC – MAP* ($\mathbb{N}$, $\mathbb{N}$) denotes the set of all the (total) recursive functions over $\mathbb{N}$.

To introduce a weaker notion of depth, it is necessary to fix the notation as to reducibilities and degrees.

Denote the *Turing reducibility* by $\leq_{\mathrm{T}}$ and the polynomial time Turing reducibility by $\leq_{\mathrm{T}}^{\mathrm{P}}$ (Odifreddi, 1989). Let us recall that there is an intermediate constrained-reducibility among them: the one, called *recursive time bound reducibility*, in which the halting-time is constrained to be not necessarily a polynomial but a generic recursive function; since *recursive time bound reducibility* may be proved to be equivalent to *truth-table reducibility* (I demand (Calude, 2002; Odifreddi, 1999) for its definition and for the proof of the equivalence) I will denote it by $\leq_{\mathrm{tt}}$.

A celebrated theorem proved by Gacs (1986) states that every sequence is computable by a Martin–Löf–Solovay–Chaitin-random sequence:

Theorem 2.1. *Gacs' Theorem:*

$$\bar{x} \leq_{\mathrm{T}} \bar{y} \ \forall \bar{x} \in \Sigma^{\infty}, \quad \forall \bar{y} \in \text{CHAITIN-RANDOM}(\Sigma^{\infty}) \tag{2.9}$$

This is no more true, anyway, if one adds the constraint of recursive time bound, leading to the following:

Definition 2.9. $\bar{x}$ is weakly deep:

$$\exists \bar{y} \in \text{CHAITIN-RANDOM}(\Sigma^{\infty}) : \neg(\bar{x} \leq_{\mathrm{tt}} \bar{y}) \tag{2.10}$$

I will denote the set of all the strongly-deep binary sequences by STRONGLY-DEEP(Σ^{∞}) and the set of all the weakly-deep binary sequences as WEAKLY-DEEP(Σ^{∞}).

Shallowness is then once more defined as the opposite of depth.

Definition 2.10. Strongly-shallow sequences of cbits:

$$\text{STRONGLY-SHALLOW}(\Sigma^{\infty}) := \Sigma^{\infty} - (\text{STRONGLY-DEEP}(\Sigma^{\infty})) \tag{2.11}$$

Definition 2.11. Weakly-shallow sequences of cbits:

$$\text{WEAKLY-SHALLOW}(\Sigma^{\infty}) := \Sigma^{\infty} - (\text{WEAKLY-DEEP}(\Sigma^{\infty})) \tag{2.12}$$

Weakly-shallow sequences of cbits may also be characterized in the following useful way (Bennett, 1988):

Theorem 2.2. *Alternative characterization of weakly-shallow sequences of cbits:*

$$\bar{x} \in \text{WEAKLY-SHALLOW}(\Sigma^{\infty}) \Leftrightarrow \exists \mu \text{ recursive} : \bar{x} \in \mu\text{-RANDOM}(\Sigma^{\infty}) \tag{2.13}$$

where, following once more the notation of Segre (2002), μ-RANDOM(Σ^{∞}) *denotes the set of all the* Martin–Löf *random sequences w.r.t. the measure* μ.

As to sequences of cbits, the considerations made in section 1 may be thoroughly formalized through the following:

Theorem 2.3. *Weak-shallowness of* Martin–Löf–Solovay–Chaitin *random sequences:*

$$\text{CHAITIN-RANDOM}(\Sigma^{\infty}) \cap \text{WEAKLY-DEEP}(\Sigma^{\infty}) = \emptyset \tag{2.14}$$

Proof: Since the Lebesgue measure μ_{Lebesgue} is recursive and by definition:

$$\text{CHAITIN-RANDOM}(\Sigma^{\infty}) = \mu_{\text{Lebesgue}} \text{ - RANDOM}(\Sigma^{\infty}) \tag{2.15}$$

the thesis immediately follows by Theorem 2.2. □

3. THE DEFINITION OF THE PHYSICAL COMPLEXITY OF CLASSICAL DYNAMICAL SYSTEMS

Since much of the fashion about complexity is based on a spread confusion among different notions, starting from the basic difference among *plain Kolmogorov complexity* K and *algorithmic information* I, much care has to be taken.

Let us start from the following notions by Brudno:

Definition 3.1. Brudno algorithmic entropy of $\bar{x} \in \Sigma^{\infty}$:

$$B(\bar{x}) := \lim_{n \to \infty} \frac{K(\vec{x}(n))}{n} \tag{3.1}$$

At this point one could think that considering the asympotic rate of *algorithmic information* instead of *plain Kolmogorov complexity* would result in a different definition of the algorithmic entropy of a sequence.

That this is not the case is the content of the following:

Theorem 3.1.

$$B(\bar{x}) = \lim_{n \to \infty} \frac{I(\vec{x}(n))}{n} \tag{3.2}$$

Proof: It immediately follows by the fact that (Staiger, 1999):

$$|I(\vec{x}(n)) - K(\vec{x}(n))| \leq o(n) \tag{3.3}$$

□

Definition 3.2. $\bar{x} \in \Sigma^{\infty}$ is Brudno-random:

$$B(\bar{x}) > 0 \tag{3.4}$$

I will denote the set of all the Brudno random binary sequences by BRUDNO(Σ^{∞}).

One great source of confusion in a part of the literature arises from the ignorance of the following basic result proved by Brudno himself (1978):

Theorem 3.2. *Brudno randomness is weaker than Chaitin randomness:*

$$\text{BRUDNO-RANDOM}(\Sigma^{\infty}) \supset \text{CHAITIN-RANDOM}(\Sigma^{\infty}) \qquad (3.5)$$

as we will see in the sequel of this section.

Following the analysis performed in Segre (2002) (to which I demand for further details), I will recall here some basic notion of Classical Ergodic Theory:

Given a classical probability space (X, μ):

Definition 3.3. Endomorphism of (X, μ):

$T : \text{HALTING}(\mu) \rightarrow \text{HALTING}(\mu)$ surjective:

$$\mu(A) = \mu(T^{-1}A) \forall A \in \text{HALTING}(\mu) \qquad (3.6)$$

where $\text{HALTING}(\mu)$ is the halting set of the measure μ, namely the σ-algebra of subsets of X on which μ is defined.

Definition 3.4. Classical dynamical system:

A triple (X, μ, T) such that

- (X, μ) is a classical probability space
- $T : \text{HALTING}(\mu) \rightarrow \text{HALTING}(\mu)$ is an endomorphism of (X, μ).

Given a classical dynamical system (X, μ, T):

Definition 3.5. (X, μ, T) is ergodic:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(A \cap T^{k}(B)) = \mu(A)\mu(B) \, \forall \, A, B \in \text{HALTING}(\mu) \qquad (3.7)$$

Definition 3.6. n-Letters alphabet:

$$\Sigma_n := \{0, \ldots, n-1\} \qquad (3.8)$$

Clearly

$$\Sigma_2 = \Sigma \qquad (3.9)$$

Given a classical probability space (X, μ):

Definition 3.7. Finite measurable partition of (X, μ):

$$\begin{gathered} A = \{A_0, \ldots A_{n-1}\} n \in \mathbb{N}: \\ A_i \in \text{HALTING}\,(\mu) \quad i = 0, \ldots, n-1 \\ A_i \cap A_j = \emptyset \, \forall i \neq \quad j \\ \mu\left(X - \cup_{i=0}^{n-1} A_i\right) = 0 \end{gathered} \qquad (3.10)$$

I will denote the set of all the finite measurable partitions of (X, μ) by $\mathcal{P}(X, \mu)$.

Given two partitions $A = \{A_i\}_{i=0}^{n-1}$, $B = \{B_j\}_{j=0}^{m-1} \in \mathcal{P}(X, \mu)$:

Definition 3.8. A is a coarse-graining of B $(A \preceq B)$: every atom of A is the union of atoms by B.

Definition 3.9. Coarsest refinement of $A = \{A_i\}_{i=0}^{n-1}$ and $B = \{B_j\}_{j=0}^{m-1} \in \mathcal{P}(X, \mu)$:

$$A \vee B \in \mathcal{P}(X, \mu)$$

$$A \vee B := \{A_i \cap B_j \quad i = 0, \ldots, n-1 \quad j = 0, \cdots, m-1\} \tag{3.11}$$

Clearly $\mathcal{P}(X, \mu)$ is closed both under coarsest refinements and under endomorphisms of (X, μ).

Let us observe that, beside its abstract, mathematical formalization, Definition 3.7 has a precise operational meaning.

Given the classical probability space (X, μ), let us suppose to make an experiment on the probabilistic universe it describes using an instrument whose distinguishing power is limited in that it is not able to distinguish events belonging to the same atom of a partition $A = \{A_i\}_{i=0}^{n-1} \in \mathcal{P}(X, \mu)$.

Consequentially the outcome of such an experiment will be a number:

$$r \in \Sigma_n \tag{3.12}$$

specifying the observed atom A_r in our coarse-grained observation of (X, μ).

I will call such an experiment an *operational observation of* (X, μ) *through the partition A*.

Considering another partition $B = \{B_j\}_{j=0}^{m-1} \in \mathcal{P}(X, \mu)$, we have obviously that the operational observation of (X, μ) through the partition $A \vee B$ is the conjuction of the two experiments consisting in the operational observations of (X, μ) through the partitions, respectively, A and B.

Consequentially we may consistently call an *operational observation of* (X, μ) *through the partition A* more simply an A-experiment.

The experimental outcome of an operational observation of (X, μ) through the partition $A = \{A_i\}_{i=0}^{n-1} \in \mathcal{P}(X, \mu)$ is a classical random variable having as distribution the stochastic vector

$$\begin{pmatrix} \mu(A_0) \\ \vdots \\ \mu(A_{n-1}) \end{pmatrix}$$

whose *classical probabilistic information*, i.e., its Shannon entropy, I will call the entropy of the partition A, according to the following:

Definition 3.10. entropy of $A = \{A_i\}_{i=0}^{n-1} \in \mathcal{P}(X, \mu)$:

$$H(A) := H\left(\begin{pmatrix} \mu(A_0) \\ \vdots \\ \mu(A_{n-1}) \end{pmatrix}\right) \tag{3.13}$$

It is fundamental, at this point, to observe that, given an experiment, one has to distinguish between two conceptually different concepts:

1. the *uncertainty of the experiment*, i.e., the amount of uncertainty on the outcome of the experiment before to realize it;
2. the *information of the experiment*, i.e., the amount of information gained by the outcome of the experiment.

As lucidly observed by Patrick Billingsley (1965), the fact that in Classical Probabilistic Information Theory both these concepts are quantified by the Shannon entropy of the experiment is a consequence of the following:

Theorem 3.3. *The soul of Classical Information Theory:*

$$\text{information gained} = \text{uncertainty removed} \tag{3.14}$$

Theorem 3.3 applies, in particular, as to the partition experiments we are discussing.

Let us now consider a classical dynamical system $CDS := (X, \mu, T)$.

The T-invariance of μ implies that the partitions $A = \{A_i\}_{i=0}^{n-1}$ and $T^{-1}A := \{T^{-1}A_i\}_{i=0}^{n-1}$ have equal probabilistic structure. Consequentially the A-*experiment* and the $T^{-1}A$-*experiment* are replicas, *not necessarily independent*, of the same experiment, made at successive times.

In the same way the $\vee_{k=0}^{n-1} T^{-k}A$-*experiment* is the compound experiment consisting of n repetitions $A, T^{-1}A, \ldots, T^{-(n-1)}A$ of the experiment corresponding to $A \in \mathcal{P}(X, \mu)$.

The amount of classical information per replication we obtain in this compound experiment is clearly

$$\frac{1}{n} H\big(\vee_{k=0}^{n-1} T^{-k}A\big)$$

It may be proved (Kornfeld and Sinai, 2000) that when n grows this amount of classical information acquired per replication converges, so that the following quantity

$$h(A, T) := \lim_{n\to\infty} \frac{1}{n} H\big(\vee_{k=0}^{n-1} T^{-k}A\big) \tag{3.15}$$

exists.

In different words, we can say that $h(A, T)$ gives the asymptotic rate of acquired classical information per replication of the A experiment.

We can at last introduce the following fundamental notion originally proposed by Kolmogorov for K-systems and later extended by Yakov Sinai to arbitrary classical dynamical systems (American Mathematical Society, 2000; Kolmogorov, 1993d; Kornfeld and Sinai, 2000; Sinai, 1976, 1994):

Definition 3.11. Kolmogorov–Sinai entropy of CDS:

$$h_{\mathrm{CDS}} := \sup_{A \in \mathcal{P}(X,\mu)} h(A, T) \tag{3.16}$$

By definition we have clearly that

$$h_{\mathrm{CDS}} \geq 0 \tag{3.17}$$

Definition 3.12. CDS is chaotic:

$$h_{\mathrm{CDS}} > 0 \tag{3.18}$$

Definition 3.12 shows explicitly that the concept of classical-chaos is an information-theoretic one: a classical dynamical system is chaotic if there is at least one experiment on the system that, no matter how many times we insist on repeating it, continues to give us classical information.

That such a meaning of classical chaoticity is equivalent to the more popular one as the sensible (i.e., exponential) dependence of dynamics from the initial conditions is a consequence of Pesin's Theorem stating (under mild assumptions) the equality of the Kolmogorov–Sinai entropy and the sum of the positive Lyapunov exponents.

This interrelation may be caught observing that

- if the system is chaotic, we know that there is an experiment whose repetition definitely continues to give information: such an information may be seen as the information on the initial condition that is necessary to furnish more and more with time if one wants to keep the error on the prediction of the phase-point below a certain bound;
- if the system is not chaotic, the repetition of every experiment is useful only a finite number of times, after which every further repetition does not furnish further information.

Let us now consider the issue of symbolically translating the coarse-grained dynamics following the traditional way of proceeding described in the second section of Alekseev and Yakobson (1981): given a number $n \in \mathbb{N} : n \geq 2$ let us introduce the following:

Definition 3.13. n-Adic value:
the map v_n: $\Sigma_n^\infty \mapsto [0, 1]$:

$$v_n(\bar{x}) := \sum_{i=1}^{\infty} \frac{x_i}{n^i} \tag{3.19}$$

the more usual notation:

$$(0.x_1 \ldots x_m \ldots)_n := v_n(\bar{x}) \quad \bar{x} \in \Sigma_n^\infty \tag{3.20}$$

and the following:

Definition 3.14. n-Adic nonterminating natural positional representation:
the map r_n: $[0, 1] \mapsto \Sigma_n^\infty$:

$$r_n((0.x_1 \ldots x_i \ldots)_n) := \bar{x} \tag{3.21}$$

with the nonterminating condition requiring that the numbers of the form $(0.x_1 \ldots x_i \overline{(n-1)})_n = (0. \ldots (x_i + 1)\bar{0})_n$ are mapped into the sequence $x_1 \ldots x_i \overline{(n-1)}$.

Given $n_1, n_2 \in \mathbb{N}$: $\min(n_1, n_2) \geq 2$:

Definition 3.15. Change of basis from n_1 to n_2:
the map cb_{n_1,n_2} : $\Sigma_{n_1}^\infty \mapsto \Sigma_{n_2}^\infty$:

$$cb_{n_1,n_2}(\bar{x}) := r_{n_2}(v_{n_1}(\bar{x})) \tag{3.22}$$

It is important to remark that (Calude, 2002):

Theorem 3.4. *Basis-independence of randomness:*

$$\text{RANDOM}(\Sigma_{n_2}^\infty) = cb_{n_1,n_2}(\text{RANDOM}(\Sigma_{n_1}^\infty))\, \forall n_1, n_2 \in \mathbb{N} : \min(n_1, n_2) \geq 2 \tag{3.23}$$

Considered a partition $A = \{A_i\}_{i=0}^{n-1} \in \mathcal{P}(X, \mu)$:

Definition 3.16. Symbolic translator of CDS w.r.t. A:
$\psi_A : X \to \Sigma_n$:

$$\psi_A(x) := i : x \in A_i \tag{3.24}$$

In this way one associates to each point of X the letter, in the alphabet having as many letters as the number of atoms of the considered partition, labeling the atom to which the point belongs.

Concatenating the letters corresponding to the phase point at different times, one can then codify $k \in \mathbb{N}$ steps of the dynamics:

Definition 3.17. k-Point symbolic translator of CDS w.r.t. A:
$\psi_A^{(k)} : X \to \Sigma_n^k$:

$$\psi_A^{(k)}(x) := \cdot_{j=0}^{k-1} \psi_A(T^j x) \tag{3.25}$$

and whole orbits:

Definition 3.18. Orbit symbolic translator of CDS w.r.t. A:
$\psi_A^{(\infty)} : X \to \Sigma_n^\infty$:

$$\psi_A^{(\infty)}(x) := \cdot_{j=0}^{\infty} \psi_A(T^j x) \tag{3.26}$$

The asymptotic rate of acquisition of *plain Kolmogorov complexity* of the binary sequence obtained translating symbolically the orbit generated by $x \in X$ through the partition $A \in \mathcal{P}(X, \mu)$ is clearly given by

Definition 3.19.

$$B(A, x) := B\left(cb_{\text{card}(A),2}\left(\psi_A^{\infty}(x)\right)\right) \tag{3.27}$$

We saw in Definition 3.11 that the *Kolmogorov–Sinai entropy* was defined as $K(A, x)$ computed on the more probabilistically informative A experiment; in the same way the *Brudno algorithmic entropy of* x is defined as the value of $B(A, x)$ computed on the more algorithmically informative A experiment:

Definition 3.20. Brudno algorithmic entropy of (the orbit starting from) x:

$$B_{\text{CDS}}(x) := sup_{A \in \mathcal{P}(X,\mu)} B\left(cb_{\text{card}(A),2}\left(\psi_A^{(\infty)}(x)\right)\right) \tag{3.28}$$

Demanding Brudno (1978) for further details, let us recall that, as it is natural for different approaches of studying a same object, the *probabilistic approach* and the *algorithmic approach* to Classical Information Theory are deeply linked:

the partial map $D_I : \Sigma^\star \overset{\circ}{\mapsto} \Sigma^\star$ defined by

$$D_I(\vec{x}) := \vec{x}^\star \tag{3.29}$$

is by construction a prefix-code of pure algorithmic nature, so that it would be very reasonable to think that it may be optimal only for some ad hoc probability distribution, i.e., that for a generic probability distribution P the *average code word length of* D_I *w.r.t.* P:

$$L_{D_I,P} = \sum_{\vec{x} \in \text{HALTING}(D_I)} P(\vec{x}) I(\vec{x}) \tag{3.30}$$

will not achieve the optimal bound, the Shannon information $H(P)$, stated by the cornerstone of Classical Probabilistic Information, i.e., the following celebrated:

Theorem 3.5. *Classical noiseless coding theorem:*

$$H(P) \leq L_P \leq H(P) + 1 \tag{3.31}$$

where L_P is the minimal average code word length allowed by the distribution P.

Contrary, the deep link between the *probabilistic-approach* and the *algorithmic-approach* makes the miracle: under mild assumptions about the distribution P the code D_I is optimal as it is stated by the following:

Theorem 3.6. *Link between Classical Probabilistic Information and Classical Algorithmic Information:*

HP:

$$P \text{ recursive classical probability distribution over } \Sigma^\star$$

TH:

$$\exists c_P \in \mathbb{R}_+ : 0 \leq L_{D_I,P} - H(P) \leq c_P \tag{3.32}$$

With an eye at Theorem 3.1 it is then natural to expect that such a link between *classical probabilistic information* and *classical algorithmic information* generates a link between the asymptotic rate of acquisition of *classical probabilistic information* and the asymptotic rate of acquisition of *classical algorithmic information* of the coarse-grained dynamics of CDS observed by repetitions of the experiments for which each of them is maximal.

Demanding to Brudno (1983) for further details such a reasoning, properly formalized, proves the following:

Theorem 3.7. *Brudno's theorem:*

HP:

$$\text{CDS ergodic}$$

TH:

$$h_{\text{CDS}} = B_{\text{CDS}}(x) \forall - \mu - a.e. x \in X \tag{3.33}$$

Let us now consider the *algorithmic approach to Classical Chaos Theory* strongly supported by Joseph Ford, whose objective is the characterization of the concept of chaoticity of a classical dynamical system as the algorithmic-randomness of its symbolically translated trajectories.

To require such a condition for all the trajectories would be too restrictive since it is reasonable to allow a chaotic dynamical system to have a countable number of periodic orbits.

Let us then introduce the following two notions:

Definition 3.21. CDS is strongly algorithmically chaotic:

$$\forall - \mu - a.e. x \in X, \exists A \in \mathcal{P}(X,\mu) : cb_{\text{card}(A),2}\big(\psi_{A}^{(\infty)}(x)\big) \in \text{CHAITIN-RANDOM}(\Sigma^{\infty}) \tag{3.34}$$

Definition 3.22. CDS is weak algorithmically chaotic:

$$\forall - \mu - a.e. x \in X, \exists A \in \mathcal{P}(X,\mu) : cb_{\text{card}(A),2}\big(\psi_{A}^{(\infty)}(x)\big) \in \text{BRUDNO-RANDOM}(\Sigma^{\infty}) \tag{3.35}$$

The difference between Definition 3.21 and Definition 3.22 follows by Theorem 3.2.

Clearly Theorem 3.7 implies the following:

Corollary 3.1.

$$\textit{chaoticity} = \textit{weak algorithmic chaoticity}$$

$$\textit{chaoticity} < \textit{strong algorithmic chaoticity}$$

that shows that the algorithmic approach to Classical Chaos Theory is equivalent to the usual one only in weak sense.

The plethora of wrong statements found in a part of the literature caused by the ignorance of Corollary 3.1 is anyway of little importance if compared with the complete misunderstanding of the difference existing among the concepts of *chaoticity* and *complexity* for classical dynamical systems; with this regards the analysis made in section 1. may be now thoroughly formalized introducing the following natural notions:

Definition 3.23. CDS is strongly-complex:

$$\forall - \mu - a.e. x \in X, \exists A \in \mathcal{P}(X,\mu) : cb_{\text{card}(A),2}\big(\psi_{A}^{(\infty)}(x)\big) \in \text{STRONGLY-COMPLEX}(\Sigma^{\infty}) \tag{3.36}$$

Definition 3.24. CDS is weakly-complex:

$$\forall - \mu - a.e. x \in X, \exists A \in \mathcal{P}(X,\mu) : cb_{\text{card}(A),2}\big(\psi_{A}^{(\infty)}(x)\big) \in \text{WEAKLY-COMPLEX}(\Sigma^{\infty}) \tag{3.37}$$

One has that

Theorem 3.8. *Weak-shallowness of chaotic dynamical systems:*

$$\text{CDS chaotic} \Rightarrow \text{CDS weakly-shallow}$$

Proof: The thesis immediately follows combining Theorem 2.3 with the definitions 3.23 and 3.24. □

4. THE DEFINITION OF THE PHYSICAL COMPLEXITY OF STRINGS AND SEQUENCES OF QUBITS

The idea that the physical complexity of a quantum object has to be measured in terms of a quantum analogue of Bennett's notion of *logical depth* has been first proposed by Nielsen (2002a,b).

Unfortunately, beside giving some general remark about the properties he thinks such a notion should have, Nielsen has not given a mathematical definition of it.

The first step in this direction consists, in my opinion, in considering that, such as the notion of *classical–logical-depth* belongs to the framework of Classical Algorithmic Information Theory, the notion of *quantum–logical-depth* belongs to the framework of Quantum Algorithmic Information Theory (Segre, 2002).

One of the most debated issues in such a discipline, first discussed by its father Svozil (1996) and rediscovered later by the following literature (Berthiaume *et al.*, 2001; Gacs, 2000; Manin, 1999; Segre, 2002; Vitanyi, 1999; Vitanyi, 2001), is whether the programs of the involved universal quantum computers have to be strings of cbits or strings of qubits.

As I have already noted in Segre (2002), anyway, it must be observed that, owing to the natural bijection among the computational basis $\mathcal{E}_\star$ of the *Hilbert space of qubits' strings* (notions that I am going to introduce) and $\Sigma^\star$, one can always assume that the input is a string of qubits while the issue, more precisely restated, is whether the input has (or not) to be constrained to belong to the computational basis.

So, denoting by $\mathcal{H}_2 := \mathbb{C}^2$ the one qubit's Hilbert space (endowed with its orthonormal computational basis $\mathcal{E}_2 := \{|i\rangle, i \in \Sigma\}$), denoting by $\mathcal{H}_2^{\otimes n} := \otimes_{k=0}^{n} \mathcal{H}_2$ the *n-qubits' Hilbert space* (endowed with its orthonormal computational basis $\mathcal{E}_n := \{|\vec{x}\rangle, \vec{x} \in \Sigma^n\}$), denoting by $\mathcal{H}_2^{\otimes \star} := \oplus_{n=0}^{\infty} \mathcal{H}_2^{\otimes n}$ the *Hilbert space of qubits' strings* (endowed with its orthonormal computational basis $\mathcal{E}_\star := \{|\vec{x}\rangle, \vec{x} \in \Sigma^\star\}$) and denoting by $\mathcal{H}_2^{\otimes \infty} := \otimes_{n \in \mathbb{N}} \mathcal{H}_2$ the *Hilbert space of qubits' sequences* (endowed with its orthonormal computational rigged-basis[2] $\mathcal{E}_\infty := \{|\bar{x}\rangle, \bar{x} \in \Sigma^\infty\}$), one simply assumes that, instead of being a *classical Chaitin universal computer*, *U* is a *quantum Chaitin universal computer*, i.e., a universal quantum computer whose input, following Svozil's original position on the mentioned issue, is constrained to

[2] As it should be obvious, the unusual locution *rigged-basis* I am used to adopt is simply a shortcut to denote that such a "basis" has to be intended in the mathematical sense it assumes when $\mathcal{H}_2^{\otimes \infty}$ is considered as endowed with a suitable rigging, i.e., as part of a suitable *rigged Hilbert space* $\mathcal{S} \subset \mathcal{H}_2^{\otimes \infty} \subset \mathcal{S}'$ as described in Reed and Simon (1975, 1980).

belong to $\mathcal{E}_\star$ and is such that, w.r.t. the natural bijection among $\mathcal{E}_\star$ and $\Sigma^\star$, satisfies the usual Chaitin constraint of having prefixfree halting-set.

The definition of the logical depth of a string of qubits is then straightforward: given a vector $|\psi\rangle \in \mathcal{H}_2^{\otimes\star}$ and a string $\vec{x} \in \Sigma^\star$:

Definition 4.1. Canonical program of $|\psi\rangle$:

$$|\psi\rangle^\star := \min_{<_l}\{\vec{y} \in \Sigma^\star : U(\vec{y}) = |\psi\rangle\} \tag{4.1}$$

Definition 4.2. Halting time of the computation with input $|\vec{x}\rangle$:

$$T(\vec{x}) := \begin{cases} \text{Number of computational steps after which } U \text{ halts on input } \vec{x}, & \text{if } U(\vec{x}) = \downarrow \\ +\infty, & \text{otherwise.} \end{cases} \tag{4.2}$$

Definition 4.3. Logical depth of $|\psi\rangle$ at significance level s:

$$D_s(|\psi\rangle) := \min\{T(\vec{y}) : U(\vec{y}) = |\psi\rangle, \vec{y}\ s\text{-incompressible}\} \tag{4.3}$$

Definition 4.4. $|\psi\rangle$ is t-deep at significance level s:

$$D_s(|\psi\rangle) > t \tag{4.4}$$

Definition 4.5. $|\psi\rangle$ is t-shallow at significance level s:

$$D_s(|\psi\rangle) \leq t \tag{4.5}$$

I will denote the set of all the t-deep strings of qubits as t-DEEP($\mathcal{H}_2^{\otimes\star}$).

Let us observe that a sharp distinction among *depth* and *shallowness* of qubits' strings is impossible; this is nothing but a further confirmation of the fact, so many times shown and analyzed in Segre (2001), that almost all the concepts of Algorithmic Information Theory, both Classical and Quantum, have a clear, conceptually sharp meaning only when sequences are taken into account.

The great complication concerning sequences of qubits consists in that their mathematically rigorous analysis requires to give up the simple language of Hilbert spaces passing to the more sophisticated language of *noncommutative spaces*; indeed, as extensively analyzed in Segre (2002) adopting the notion of *noncommutative cardinality* therein explicitly introduced,[3] the fact that the correct noncommutative space of qubits' sequences is the hyperfinite II_1 factor:

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[3] Following Miklos Redei's (1998, 2001) many remarks mentioned in Segre (2002), about how von Neumann considered his classification of factors as a theory of noncommutative cardinalities although he never thought, as well as Redei, that the same $\aleph$ symbolism of the commutative case could be adopted.

Definition 4.6. Noncommutative space of qubits' sequences:

$$\Sigma_{\mathrm{NC}}^{\infty} := \otimes_{n=0}^{\infty}(M_2(\mathbb{C}), \tau_{\mathrm{unbiased}}) = R \tag{4.6}$$

and not the noncommutative space $\mathcal{B}(\mathcal{H}_2{}^{\otimes\infty})$ of all the bounded linear operators on $\mathcal{H}_2{}^{\otimes\infty}$ (that could be still managed in the usual language of Hilbert spaces) is proved by the fact that, as it must be, $\Sigma_{\mathrm{NC}}^{\infty}$ has the *continuum noncommutative-cardinality*:

$$\mathrm{card}_{\mathrm{NC}}(\Sigma_{\mathrm{NC}}^{\infty}) = \aleph_1 \tag{4.7}$$

while $\mathcal{B}(\mathcal{H}_2{}^{\otimes\infty})$ has only the *countable noncommutative cardinality*:

$$\mathrm{card}_{\mathrm{NC}}(\Sigma_{\mathrm{NC}}^{\star}) = \aleph_0 \tag{4.8}$$

While Definition 2.8 of a strongly-deep sequence of cbits has no natural quantum analogue, the definition of a weakly-deep sequence of qubits is straightforward.

Denoting by RANDOM($\Sigma_{\mathrm{NC}}^{\infty}$) the space of all the algorithmically random sequences of qubits, for whose characterization I demand to (Segre, 2002), let us observe that the equality between *truth-table reducibility* and *recursive time bound reducibility* existing as to Classical Computation may be naturally imposed to Quantum Computation in the following way:

Given two arbitrary mathematical quantities x and y:

Definition 4.7. x is quantum-truth-table reducible to y:

$$x \leq_{\mathrm{tt}}^{Q} y := x \text{ is } U\text{-computable from } y \text{ in bounded } U\text{-computable time.} \tag{4.9}$$

Given a sequence of qubits $\bar{a} \in \Sigma_{\mathrm{NC}}^{\infty}$:

Definition 4.8. $\bar{a}$ is weakly-deep:

$$\exists \bar{b} \in \mathrm{RANDOM}(\Sigma_{\mathrm{NC}}^{\infty}) : \neg\big(\bar{a} \leq_{\mathrm{tt}}^{Q} \bar{b}\big) \tag{4.10}$$

Denoting the set of all the weakly-deep sequences of qubits as WEAKLY-DEEP($\Sigma_{\mathrm{NC}}^{\infty}$):

Definition 4.9. Set of all the weakly-shallow sequences of qubits:

$$\text{WEAKLY-SHALLOW}(\Sigma_{\mathrm{NC}}^{\infty}) := \Sigma_{\mathrm{NC}}^{\infty} - (\text{WEAKLY-DEEP}(\Sigma_{\mathrm{NC}}^{\infty})) \tag{4.11}$$

It is natural, at this point, to conjecture that an analogue of Theorem 2.2 exists in Quantum Algorithmic Information Theory too.

Conjecture 4.1. *Alternative characterization of weakly-shallow sequences of qubits:*

$$\bar{a} \in \text{WEAKLY-SHALLOW}(\Sigma_{\mathrm{NC}}^{\infty}) \Leftrightarrow \exists \omega \in S(\Sigma_{\mathrm{NC}}^{\infty})U\text{-computable} :$$

$$\bar{a} \in \omega\text{-RANDOM}(\Sigma_{NC}^{\infty}) \tag{4.12}$$

where ω-RANDOM(Σ_{NC}^{∞}) *denotes the set of all the* ω*-random sequences of qubits w.r.t. the state* $\omega \in S(\Sigma_{NC}^{\infty})$ *to be defined generalizing the definition of* RANDOM(Σ_{NC}^{∞}) *to states different by* τ_{unbiased} *along the lines indicated in* Segre (2001) *as to the definition of the laws of randomness* $\mathcal{L}_{\text{RANDOMNESS}}^{\text{NC}}(\Sigma_{NC}^{\infty}, \omega)$ *of the noncommutative probability space* $(\Sigma_{NC}^{\infty}, \omega)$.

As to sequences of qubits, the considerations made in section 1 may be thoroughly formalized, at the prize of assuming Conjecture 4.1 as an hypothesis, through the following:

Theorem 4.1. *Weak-shallowness of random sequences of qubits:*
HP:

Conjecture 4.1 holds

TH:

$$\text{RANDOM}\left(\Sigma_{NC}^{\infty}\right) \cap \text{WEAKLY} - \text{DEEP}\left(\Sigma_{NC}^{\infty}\right) = \emptyset \tag{4.13}$$

Proof: Since the *unbiased state* τ_{unbiased} is certainly U-computable and by definition:

$$\text{RANDOM}(\Sigma_{NC}^{\infty}) = \tau_{\text{unbiased}}\text{-RANDOM}(\Sigma_{NC}^{\infty}) \tag{4.14}$$

the assumption of Conjecture 4.1 as an hypothesis immediately leads to the thesis. □

5. THE DEFINITION OF THE PHYSICAL COMPLEXITY OF QUANTUM DYNAMICAL SYSTEMS

As we have seen in section 3 the Kolmogorov–Sinai entropy $h_{\text{KS}}(CDS)$ of a *classical dynamical system* $CDS := (X, \mu, T)$ has a clear physical information-theoretic meaning that we can express in the following way:

1. an experimenter is trying to obtain information about the dynamical evolution of CDS performing repeatedly on the system a given experiment exp $\in$ EXPERIMENTS;
2. $h(\text{exp}, CDS)$ is the asymptotic rate of acquisition of classical information about the dynamics of CDS that he acquires replicating exp;
3. $h_{\text{KS}}(CDS)$ is such an asymptotic rate, computed for the more informative possible experiment:

$$h_{\text{KS}}(CDS) = \sup_{\text{exp} \in \text{EXPERIMENTS}} h(\text{exp}, CDS) \tag{5.1}$$

Let us now pass to analyze quantum dynamical systems, for whose definition and properties I demand to (Segre, 2002).

Given a *quantum dynamical system* (*QDS*) the physical information-theoretical way of proceeding would consist in analyzing the same experimental situation in which an experimenter is trying to obtain information about the dynamical evolution of *QDS* performing repeatedly on the system a given experiment $\exp \in$ EXPERIMENTS:

1. to define $h(\exp, QDS)$ as the asymptotic rate of acquisition of information about the dynamics of *QDS* that he acquires replicating the experiment exp;
2. to define the *dynamical entropy* of *QDS* as such an asymptotic rate, computed for the more informative possible experiment:

resulting in the following:

Definition 5.1. Dynamical entropy of *QDS*:

$$h_{\text{d.e.}}(QDS) = \sup_{\exp \in \text{EXPERIMENTS}} h(\exp, QDS) \tag{5.2}$$

Definition 5.2. *QDS* is chaotic:

$$h_{\text{d.e.}}(QDS) > 0 \tag{5.3}$$

The irreducibility of Quantum Information Theory to Classical Information Theory, caused by the fact that Theorem 3.3 does not extend to the quantum case owing to the existence of some nonaccessible information about a quantum system (as implied by the Grönwald–Lindblad–Holevo Theorem) and the consequent irreducibility of the qubit to the cbit (Nielsen and Chuang, 2000; Segre, 2002), would then naturally lead to the physical issue whether the information acquired by the experimenter is classical or quantum, i.e., if $h_{\text{d.e.}}$ (*QDS*) is a number of cbits or a number of qubits.

Such a physical approach to quantum dynamical entropy was performed first by Lindblad (1979) and later refined and extended by Robert Alicki and Mark Fannes resulting in the so-called *Alicki–Lindblad–Fannes entropy* (Alicki and Fannes, 2001).

Many attempts to define a quantum analogue of the *Kolmogorov–Sinai entropy* pursued, instead, a different purely mathematical approach consisting in generalizing noncommutatively the mathematical machinery of partitions and coarsest refinements underlying Definition 3.11, obtaining mathematical objects whose (eventual) physical meaning was investigated subsequently.

This was certainly the case as to the *Connes–Narnhofer–Thirring entropy*, the *entropy of Sauvageot and Thouvenot* and *Voiculescu's approximation entropy* (Connes *et al.*, 1998; Stormer, 2000).

As to the *Connes–Narnhofer–Thirring entropy*, in particular, the noncommutative analogue playing the role of the classical partitions are the so-called *Abelian models* whose (eventual) physical meaning is rather obscure since, as it has been lucidly shown by Fabio Benatti in his very beautiful book (Benatti, 1993), they do not correspond to physical experiments performed on the system, since even a projective measurement (i.e., a measurement corresponding to a Projection Valued Measure) cannot, in general, provide an abelian model, owing to the fact that its reduction formula corresponds to a decomposition of the state of the system if and only if the measured observable belongs to the centralizer of the state of the system.

It may be worth observing, by the way, that the nonexistence of an agreement into the scientific community as to the correct quantum analogue of the *Kolmogorov–Sinai entropy* and hence on the definition of *quantum chaoticity* should not surprise, such an agreement lacking even for the well more basic notion of *quantum ergodicity*, *Zelditch's quantum ergodicity* (Zelditch, 1996) (more in the spirit of the original *Von Neumann's quantum ergodicity* (von Neumann, 1929) to which it is not anyway clear if it reduces exactly as to quantum dynamical systems of the form (A, ω, α) with $\text{card}_{\text{NC}}(A) \leq \aleph_0$ and $\alpha \in \text{INN}(A)$) differing from *Thirring's quantum ergodicity* (Thirring, 1983) adopted both in Benatti (1993) and in (Alicki and Fannes, 2001).

Returning, now, to the physical approach based on Definition 5.1, the mentioned issue whether the dynamical entropy $h_{\text{d.e.}}(QDS)$ is a measure of *classical information* or of *quantum information* (i.e., if it is a number of cbits or qubits) is of particular importance as soon as one tries to extend to the quantum domain Joseph Ford's algorithmic approach to Chaos Theory seen in section 3:

1. in the former case, in fact, one should define *quantum algorithmic chaoticity* by the requirement that almost all the trajectories, symbolically codified in a suitable way, belong to $\text{BRUDNO}(\Sigma^{\infty})$ for *quantum weak algorithmic chaoticity* and to $\text{CHAITIN-RANDOM}(\Sigma^{\infty})$ for *quantum strong algorithmic chaoticity*;
2. in the latter case, instead, one should define *quantum algorithmic chaoticity* by the requirement that almost all the trajectories, symbolically codified in a suitable way, belong to $\text{RANDOM}(\Sigma_{\text{NC}}^{\infty})$.

In any case one would then be tempted to conjecture the existence of a Quantum Brudno's Theorem stating the equivalence of *quantum chaoticity* and *quantum algorithmic chaoticity*, at least in weak sense, for *quantum ergodic dynamical systems*.

The mentioned issue whether the dynamical entropy $h_{\text{d.e.}}(QDS)$ is a measure of *classical information* or of *quantum information* (i.e., if it is a number of cbits or qubits) is of great importance also as to the definition of a *deep quantum dynamical system* (i.e., a *physically-complex quantum dynamical system*):

1. in the former case, in fact, one should define a *strongly (weakly)-deep quantum dynamical system* as a quantum dynamical system such that almost all its trajectories, symbolically codified in a suitable way, belong to STRONGLY-DEEP(Σ^{∞})(WEAKLY-DEEP(Σ^{∞}));
2. in the latter case, instead, one should define a *weakly-deep quantum dynamical system* as a quantum dynamical system such that almost all its trajectories, symbolically codified in a suitable way, belong to WEAKLY-DEEP($\Sigma^{\infty}_{\mathrm{NC}}$).

In any case, or by Theorem 2.3 or by Theorem 4.1, one would be almost certainly led to a quantum analogue of Theorem 3.8 stating that a *chaotic quantum dynamical system* is *weakly-shallow*, i.e., is not *physically complex*.

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